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Technical Note

The role of longitudinal diffusion in fully developed forced convective slug flow in a channel

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1. Introduction

It appears that the fullest exposition currently available of fully developed forced convection in a channel is that in Bejan [1]. However, the discussion there is incomplete in two respects. First, although the mean (bulk) temperature, T_m, is clearly defined as a velocity-weighted average over the channel cross section, it is not made explicit how T_m is determined in a particular circumstance. Second, and more importantly, at an early stage of the discussion, Bejan makes the assumption that the Péclet number is large and so the axial (longitudinal) conduction term is negligible in comparison with the transverse (radial or cross-channel) conduction term in the energy equation. A scaling analysis then leads to the conclusion that, if thermal boundary layers are merged (i.e., there is thermally fully developed flow), the Nusselt number, *Nu*, is a constant of order 1, and as a result the temperature distribution is such that $(T_w - T)/(T_w - T_m)$ is a function of the transverse coordinate only. (Here T denotes the fluid temperature, T_w denotes the wall temperature and it is assumed that the velocity profile is fully developed.)

As a result the reader is left with the impression that negligible axial conduction is a necessary assumption for the formation of a fully developed temperature profile and consequently that the Nusselt number is independent of the axial coordinate. The main purpose of this note is to point out that this is not necessarily so for one particular velocity distribution, namely slug flow. (The slug flow profile is appropriate to a porous medium when Darcy's Law is valid, and it is also appropriate for the hydrodynamically undeveloped flow of a fluid of low Prandtl number.) The question of determining T_m is also addressed.

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2. Analysis, results and discussion

We consider the forced convection flow of a Newtonian incompressible fluid (with constant properties) along a regular (constant cross-section area) channel with the same two sorts of thermal boundary conditions as did Bejan [1], namely uniform wall heat flux or uniform wall temperature. We now demonstrate that for the cases of uniform wall heat flux and uniform wall temperature the effect of axial conduction on *Nu* is zero, provided that the velocity profile is one of slug flow.

2.1. Case A—uniform wall heat flux

It has been widely reported (see, for example, Ref. [2]) that in the case of uniform heat flux the effect of axial conduction on the value of *Nu* is zero, a direct consequence of zero axial temperature gradient, but without any precise explanation of why this is so. The result does not follow solely from an application of the First Law of Thermodynamics applied to the bulk flow.

In fact, the First Law of Thermodynamics, as applied to a channel section of area A and infinitesimal length dx, requires that the axial enthalpy variation equals the total heat crossing the channel boundaries, and therefore

$$\rho U A c_p \frac{dT_m}{dx} = pq'' + A\frac{dq''_{cd}}{dx} \quad (1)$$

where ρ is the fluid density, U is in general the fluid average velocity, A is the cross-section area of the channel, c_p is the fluid specific heat at constant pressure, T_m is the mean bulk fluid temperature, p is the perimeter of the channel, q'' is the solid boundary heat flux, and q''_{cd} is the heat diffused by conduction through the fluid across A (constant A and k) a parameter related to the fluid cross-section averaged temperature T_A via the definition

$$q''_{cd} = \frac{1}{A}\int_A -k\frac{dT}{dx}dA = -k\frac{d}{dx}\left(\frac{1}{A}\int_A T\,dA\right) = -k\frac{dT_A}{dx}. \tag{2}$$

The last term of equation (1), i.e., the longitudinal diffusion term, is routinely neglected from the analysis. Observe that equation (2) defines T_A. In the case of slug flow $T_A = T_m$ and equation (1) reduces to

$$\frac{d^2T_m}{dx^2} + \left(\frac{U}{\alpha}\right)\frac{dT_m}{dx} = \left(\frac{p}{kA}\right)q'' \tag{3}$$

where α is the thermal diffusivity of the fluid. Equation (3) can be easily solved for a channel with isoflux boundary condition (uniform q'') using the variation of parameters technique [3]. The mean-temperature solution is

$$T_m(x) = \left(\frac{pq''}{\rho c_p AU}\right)x + C_1\left(\frac{\alpha}{U}\right)e^{-\frac{Ux}{\alpha}} + C_2 \tag{4}$$

where C_1 and C_2 are two arbitrary constants to be determined using two conditions at $x = 0$, the start of the heating section. These conditions are the values of the cross-section averaged temperature equal to T_0, and the zero longitudinal temperature gradient. Equation (4) can then be rewritten as,

$$T_m(x) = T_0 + \left(\frac{pq''}{\rho Ac_p U}\right)\left\{x + \frac{\alpha}{U}\left[e^{-\frac{U}{\alpha}x} - 1\right]\right\}. \tag{5}$$

Therefore, the First Law indicates only that T_m is the sum of a term which increases linearly, and one which decreases exponentially, in the longitudinal direction when longitudinal diffusion is not neglected. We can show now that for the case of slug flow, the Nusselt number is a constant, i.e., unaffected by the longitudinal diffusion even when longitudinal diffusion is present. We start by considering $T_m = T_A$ (slug flow) and the definition of T_A, equation (2),

$$\frac{dT_m}{dx} = \frac{1}{A}\int_A \frac{\partial T}{\partial x}dA. \tag{6}$$

When U is a constant, the differential equation expressing the conservation of energy (e.g., equation (8) for one case) is such that the axial and transverse variables can be separated. It follows that $\partial T/\partial x$ can be written in the form $f(x)g_1(\xi,\eta)$, where (ξ,η) are the coordinates in the transverse place. Then equation (6) implies that dT_m/dx is of the form $Cf(x)$, where C is a constant resulting from the integration of $g_1(\xi,\eta)$ on A. Thus $\partial T/\partial x = [g_1(\xi,\eta)/C]dT_m/dx = g_2(\xi,\eta)dT_m/dx$. Since the flow is assumed to be thermally fully developed, $g_2(\xi,\eta) = 1$. (If this was not so, then the difference between T and T_m would grow without bound as x increases.) Since T_w is just T at a particular (ξ,η) location, it follows that

$$\frac{\partial(T - T_m)}{\partial x} = \frac{d(T_w - T_m)}{dx} = 0. \tag{7}$$

Finally, using the Nusselt number definition $Nu \propto q''/(T_w - T_m)$, one deduces the Nu is constant for uniform heat flux and slug flow even when longitudinal diffusion is not neglected.

2.2. Case B—uniform wall temperature

We now demonstrate that for the case of uniform wall temperature the effect of axial conduction on Nu is also zero, provided that the velocity profile is one of slug flow. This is so because, although the boundary heat flux q'' and the temperature difference $T_w - T_m$ each depend on the axial coordinate x, their ratio is independent of x. In order to be explicit, we now consider separately the cases of flow between parallel flat plates and flow in a circular cylinder.

2.3. Case B1—parallel plates

Suppose that the plates are at $y = \pm H$ and the uniform velocity is U in the x-direction. The pointwise energy equation for the temperature T is

$$\frac{U}{\alpha}\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}, \tag{8}$$

where α is the thermal diffusivity, and the boundary conditions are

$$T = T_w \quad \text{at} \quad y = \pm H. \tag{9}$$

The method of separation of variables leads to the solution

$$T - T_w = X(x)\cos(\pi y/2H) \tag{10}$$

where

$$X(x) = e^{Ux/2\alpha}(Ae^{\beta x} + Be^{-\beta x}) \tag{11}$$

A and B are arbitrary constants, and

$$\beta = (1 + \pi^2\alpha^2/4H^2U^2)^{1/2}. \tag{12}$$

The bulk mean temperature is thus

$$T_m = T_w + (2/\pi)X(x). \tag{13}$$

The Nusselt number is defined as

$$Nu = \frac{2H}{k}\frac{q''}{(T_w - T_m)} \tag{14}$$

where

$$q'' = k(\partial T/\partial y)_{y=H} = (\pi k/2H)X(x), \tag{15}$$

and k is the thermal conductivity of the fluid. Substitution of equations (13) and (15) into equation (14) then gives

$$Nu = \pi^2/2 = 4.93, \tag{16}$$

independent of the value of x and independent of the

Peclét number. This value is reasonable well known; see, for example, Ref. [4].

In order to determine the axial temperature distribution, the constants A and B must be found. This can be done using equation (13) if the value of T_m is measured at two values of x.

2.4. Case B2—circular cylinder

Let R be the radius of the cylinder. In place of equation (9) one now has

$$T = T_w \quad \text{at} \quad r = R, \tag{17}$$

and in place of equation (10) one finds that

$$T = T_w + X(x)J_0(\lambda r), \tag{18}$$

where $X(x)$ is still given by equation (11) and, in order to satisfy equation (17),

$$\lambda R = 2.40482 \tag{19}$$

this being the first zero of the Bessel function of the first kind of order zero, $J_0(z)$. Using standard results of Bessel function, it is then found that

$$T_m = T_w + 2X(x)J_1(\lambda R)/\lambda R. \tag{20}$$

The Nusselt number is given by

$$Nu = 2R[\partial T/\partial r]_{r=R}/(T_w - T_m) \tag{21}$$

and when use is made of equations (20) and (21) one obtains

$$Nu = (\pi R)^2 = 5.78. \tag{22}$$

The same result for Nu can also be obtained directly by observing that the equation analogous to equation (3.72) of Bejan [1] can be written as Bessel's equation of order zero in the independent variable $Nu^{1/2}(r/R)$. Again, Nu is independent of the x-coordinate and independent of the Péclet number. The value 5.78 agreed with that given by Bejan [5], who referred at this point to Rohsenow and Choi [6], who in fact give the less accurate value 5.75, while Kays and Harnett [4] list the value 5.80. None of these authors gives information about the temperature distribution.

In summary, when one has a uniform velocity profile (slug flow), the value of the Nusselt number is not affected by finite axial conduction, for the case of uniform heat flux or uniform temperature boundary conditions. When the velocity varies with the transverse coordinate the situation will be otherwise.

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